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Statistics of the occupation time for a random walk in the presence of a moving boundary

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Abstract
We investigate the distribution of the time spent by a random walker to the right of a boundary moving with constant velocity v. For the continuous-time problem (Brownian motion), we provide a simple alternative proof of Newman's recent result (Newman T J 2001 *J. Phys. A: Math. Gen.* **34** L89) using a method developed by Kac. We then discuss the same problem for the case of a random walk in discrete time with an arbitrary distribution of steps, taking advantage of the general set of results of Sparre Andersen. For the binomial random walk we analyse the corrections to the continuum limit on the example of the mean occupation time. The case of Cauchy-distributed steps is also studied.

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1. Introduction

Consider a Brownian particle, starting from the origin, whose position x_t satisfies the Langevin equation

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = \eta_t$$

where η_t is Gaussian white noise, such that $\langle \eta_t \rangle = 0$ and $\langle \eta_t \eta_{t'} \rangle = 2D\,\delta(t - t')$.

In a recent paper [1], Newman considered the following question: what is the distribution of the length of time spent by the particle to the right of a boundary moving with constant velocity v? This quantity, denoted by T_t^+, is known as the occupation time of the half-line located to the right of the boundary. It reads

$$T_t^+ = \int_0^t \mathrm{d}t'\, I_{t'} \qquad I_{t'} = \Theta(x_{t'} - vt')$$

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where $\Theta(x)$ is the Heaviside function. The indicator random variable I_t is therefore equal to unity if $x_t > vt$, and zero otherwise. Similarly, the occupation time to the left of the moving boundary is denoted by T_t^-, such that $T_t^+ + T_t^- = t$.

A number of past studies as well as more recent ones have been devoted to the statistics of the occupation time of stochastic processes, either in probability theory [2–5], or in statistical physics [6–15] in the context of persistence.

A derivation of the probability density of T_t^+, $f_{T_t^+}(t,\tau) = \mathrm{d}\mathcal{P}(T_t^+ < \tau)/\mathrm{d}\tau$, is given in [1], with the result

$$f_{T_t^+}(t,\tau) = F^+(\tau)F^-(t-\tau) \tag{1}$$

where

$$F^\pm(\tau) = \frac{1}{\sqrt{\pi\tau}} \exp\left(-\frac{v^2\tau}{4D}\right) \mp \frac{v}{2\sqrt{D}} \operatorname{erfc}\left(\pm\frac{v}{2}\sqrt{\frac{\tau}{D}}\right) \tag{2}$$

and erfc is the complementary error function.

In particular, in the case of a static boundary ($v = 0$), we have

$$f_{T_t^+}(t,\tau) = \frac{1}{\pi}\frac{1}{\sqrt{\tau(t-\tau)}} \tag{3}$$

hence the fraction of time T_t^+/t spent by the Brownian particle to the right of the origin admits a limiting distribution as $t \to \infty$, which reads

$$\lim_{t\to\infty} f_{T_t^+/t}(x) = \frac{1}{\pi\sqrt{x(1-x)}} \qquad (0 < x < 1). \tag{4}$$

The arcsine law [2, 16] is thus recovered.

The aim of this paper is to complement Newman's work in two directions.

Firstly, we give an alternative, simpler derivation of equations (1), (2), using a method originally developed by Kac [3].

Secondly, we discuss the corresponding problem for a sum of random variables, i.e. for a random walk in discrete time, with an arbitrary distribution of steps, either narrow or broad. The general set of results of Sparre Andersen [16, 17] is the starting point of this analysis. A factorization property of the distribution of the occupation time similar to (1) holds, the role of $F^+(\tau)$ being played by the quantity F_k^+, which is simply the survival probability of the walk in the presence of the boundary. We investigate two examples in more detail. For the binomial random walk, for which a detailed study of F_k^+ can be found in [18], we analyse the corrections to the continuum limit on the example of the mean occupation time; for a Cauchy distribution of steps, we determine the probability distribution of the occupation time.

2. Brownian motion

The problem of a particle executing symmetric Brownian motion in the presence of a boundary moving with constant velocity v is equivalent to that of biased Brownian motion with velocity $-v$ in the presence of a fixed boundary, located at the origin. Consider the probability for this biased Brownian walk to be at position x at time t, and to have spent a length of time equal to τ to the right of the origin. The joint probability density of the event $(x_t = x, T_t^+ = \tau)$ is denoted by $p(t,\tau,x)$. We then have

$$f_{T_t^+}(t,\tau) = \int_{-\infty}^{\infty} \mathrm{d}x\; p(t,\tau,x). \tag{5}$$

The method of Kac[4] consists in writing a master equation for $p(t, \tau, x)$. It is an easy matter to realize that the latter reads, in the present case,

$$\frac{\partial p}{\partial t} + \Theta(x)\frac{\partial p}{\partial \tau} = D\frac{\partial^2 p}{\partial x^2} + v\frac{\partial p}{\partial x} \tag{6}$$

with initial condition $p(0, \tau, x) = \delta(x)\,\delta(\tau)$. In Laplace space, setting

$$\hat{p}(s, u, x) = \underset{t,\tau}{\mathcal{L}}\ p(t, \tau, x)$$

equation (6) yields

$$(s + u\Theta(x))\,\hat{p} - D\frac{\partial^2 \hat{p}}{\partial x^2} - v\frac{\partial \hat{p}}{\partial x} = \delta(x). \tag{7}$$

This inhomogeneous differential equation is easily solved. By requiring $\hat{p}(s, u, x)$ to vanish at infinity ($x \to \pm\infty$), we have

$$\hat{p}(s, u, x) = A(s, u) \times \begin{cases} \exp\left[-\left(v + \sqrt{v^2 + 4D(s+u)}\right)\dfrac{x}{2D}\right] & (x \geqslant 0) \\ \exp\left[\left(-v + \sqrt{v^2 + 4Ds}\right)\dfrac{x}{2D}\right] & (x \leqslant 0). \end{cases}$$

The amplitude $A(s, u)$ is determined by the right-hand side of (7), yielding the condition

$$\frac{\partial \hat{p}(s, u, 0^+)}{\partial x} - \frac{\partial \hat{p}(s, u, 0^-)}{\partial x} = -\frac{1}{D}$$

hence

$$A(s, u) = \frac{\sqrt{v^2 + 4D(s+u)} - \sqrt{v^2 + 4Ds}}{2Du}.$$

Finally equation (5) yields

$$\hat{f}_{T_t^+}(s, u) = \underset{t}{\mathcal{L}}\left\langle \mathrm{e}^{-uT_t^+}\right\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\ \hat{p}(s, u, x) = \hat{F}^+(s+u)\hat{F}^-(s) \tag{8}$$

with

$$\hat{F}^{\pm}(u) = \frac{2\sqrt{D}}{\sqrt{v^2 + 4Du} \pm v}. \tag{9}$$

These functions are precisely the Laplace transforms with respect to τ of $F^{\pm}(\tau)$ given in equation (2). An inverse Laplace transformation finally leads to the result (1).

Let us now comment on the form of the solution (1). A first comment concerns symmetry. It is clear that, for a given velocity v, $T_t^+(v)$ has the same distribution as $T_t^-(-v)$. Therefore

$$f_{T_t^+(-v)}(t, \tau) = f_{T_t^-(v)}(t, \tau) = f_{T_t^+(v)}(t, t - \tau).$$

This requirement is satisfied by the solution (1), since changing v in $-v$ changes $F^{\pm}(\tau)$ into $F^{\mp}(\tau)$, according to (2), or equivalently changes $\hat{F}^{\pm}(u)$ into $\hat{F}^{\mp}(u)$, according to (9). A second comment is on normalization. Integrating equation (1) upon $\tau \in [0, t]$ yields

$$1 = \mathcal{P}(T_t^+ < t) = \int_0^t \mathrm{d}\tau\ f_{T_t^+}(t, \tau) = F^+(t) * F^-(t)$$

the star in the right-hand side denoting a convolution product. Hence in Laplace space the equality $\hat{F}^+(s)\hat{F}^-(s) = 1/s$ should hold. This is indeed the case, as can be seen from (9).

[4] This method has recently been applied to the investigation of the distribution of the occupation time of subordinated Brownian motion [13] (see also [14]).

For $v = 0$, the distribution of the occupation time is given by (3). The first correction to this behaviour for small v is given by

$$f_{T_t^+}(t,\tau) = \frac{1}{\pi}\frac{1}{\sqrt{\tau(t-\tau)}} + \frac{v}{2\sqrt{\pi D}}\left(\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{t-\tau}}\right) + \cdots.$$

As a consequence,

$$\langle T_t^+\rangle = \frac{t}{2}\left(1 - \frac{2v}{3}\sqrt{\frac{t}{\pi D}} + \cdots\right).$$

This expansion, valid for small v, is singular at long times. Hence the two limits $v \to 0$ and $t \to \infty$ do not commute, showing that the presence of any bias $v \neq 0$ is relevant in the long-time regime. There is actually a non-trivial limiting distribution for T_t^+ as $t \to \infty$ if $v > 0$ (and, by symmetry, for T_t^- if $v < 0$). We have indeed $\lim_{t\to\infty} F^-(t) = \lim_{s\to 0} s\hat{F}^-(s) = v/\sqrt{D}$, so that, by (1),

$$f_{T^+}(\tau) = \frac{v}{\sqrt{D}}\, F^+(\tau) \tag{10}$$

with the notation $T^+ = \lim_{t\to\infty} T_t^+$. Using the asymptotic expansion

$$\operatorname{erfc}(x) = \frac{\mathrm{e}^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \cdots\right)$$

we see that the distribution (10) falls off exponentially for large τ, as

$$f_{T^+}(\tau) \approx \frac{2}{v}\sqrt{\frac{D}{\pi\tau^3}}\,\exp\left(-\frac{v^2\tau}{4D}\right)$$

so that all the moments $\langle (T^+)^n\rangle$ are finite. The latter can be computed by noting that

$$\hat{f}_{T^+}(u) = \frac{v}{\sqrt{D}}\,\hat{F}^+(u)$$

which, expanded around $u = 0$, leads to

$$\langle (T^+)^n\rangle = \frac{(2n)!}{(n+1)!}\left(\frac{D}{v^2}\right)^n \qquad (n \geqslant 1). \tag{11}$$

3. Random walk in discrete time

Consider a discrete random walk defined by a sum of independent, identically distributed random variables:

$$x_n = \sum_{i=1}^{n} \eta_i$$

with an arbitrary distribution of the steps η_i (either discrete or continuous, narrow or broad). For this random walk, the occupation time to the right of the boundary moving with velocity v is defined as

$$T_n^+ = \sum_{m=1}^{n} I_m \qquad I_m = \Theta(x_m - vm)$$

hence the indicator random variable $I_m = 1$ if $x_m > vm$, or 0 otherwise. The occupation time T_n^- to the left of the boundary is defined likewise, and such that $T_n^+ + T_n^- = n$. As above, we note the equivalence of the problem thus stated with that of the occupation time of a biased random walk with steps $\eta_i - v$ in the presence of a fixed boundary, located at the origin.

For sums of independent random variables, a result due to Sparre Andersen [16, 17] expresses the probability distribution of T_n^+ as the product

$$\mathcal{P}(T_n^+ = k) = \mathcal{P}(T_k^+ = k)\mathcal{P}(T_{n-k}^- = n-k). \tag{12}$$

In this equation, $\mathcal{P}(T_k^+ = k)$, hereafter denoted by F_k^+, is the probability that the walk remains to the right of the boundary up to time k, or

$$F_k^+ = \mathcal{P}(T_k^+ = k) = \langle I_1 I_2 \ldots I_k \rangle.$$

Similarly,

$$F_k^- = \mathcal{P}(T_k^- = k) = \langle (1-I_1)(1-I_2)\ldots(1-I_k) \rangle$$

is the probability that the random walker remains to the left of the boundary up to time k. In other words, the quantities $F_k^\pm$ are survival probabilities of the walk in the presence of the boundary, up to time k [6–9, 18, 19]. For instance,

$$F_k^+ = \mathcal{P}\big(x_m > vm \text{ for } 1 \leqslant m \leqslant k\big).$$

The generating function of the F_n^+ is related to the generating function of the one-time probabilities $\langle I_n \rangle = \mathcal{P}(x_n > vn)$ by [16, 17]

$$\sum_{n=0}^{\infty} F_n^+ z^n = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \langle I_n \rangle\right). \tag{13}$$

Equation (12) is the discrete counterpart of equation (1). Together with (13), it provides the answer to the question posed (in terms of the one-time quantities $\langle I_n \rangle$).

We now illustrate the above formalism by two examples. First, for a narrow distribution of steps η_i, we determine the continuum limit of equations (12) and (13), thus recovering the results (1) and (2) obtained for Brownian motion. For the binomial random walk, we investigate the corrections to the continuum limit on the example of the mean occupation time. Then, for a Cauchy distribution of steps, we determine the distribution of the occupation time $\mathcal{P}(T_n^+ = k)$ explicitly.

Consider a narrow distribution of steps, with $\langle \eta \rangle = 0$ and $\langle \eta^2 \rangle = \sigma^2$. The continuum limit is defined as $n \to \infty$, $v \to 0$, with $v\sqrt{n}/\sigma = \xi$ fixed. The central limit theorem yields

$$\langle I_n \rangle = \mathcal{P}(x_n > vn) = \mathcal{P}\left(\frac{x_n}{\sigma\sqrt{n}} > \xi\right) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} \mathrm{d}u\, \mathrm{e}^{-u^2/2} = \frac{1}{2}\left(1 - \operatorname{erf}\frac{\xi}{\sqrt{2}}\right). \tag{14}$$

Let us analyse equation (13) in the same limit. Setting $z = \mathrm{e}^{-u}$, to leading order as $u \to 0$, identifying generating series with Laplace transforms, we obtain the following estimates:

$$\sum_{n=0}^{\infty} F_n^+ z^n \approx \hat{F}^+(u) \qquad \sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1-z) \approx -\ln u$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{erf}\frac{v\sqrt{n}}{\sigma\sqrt{2}} \approx \int_0^{\infty} \frac{\mathrm{d}t}{t} \mathrm{e}^{-ut} \operatorname{erf}\frac{v\sqrt{t}}{\sigma\sqrt{2}} = 2\ln\left(\frac{v}{\sigma\sqrt{2u}} + \sqrt{\frac{v^2}{2\sigma^2 u} + 1}\right).$$

Hence finally

$$\hat{F}^+(u) = \frac{\sigma\sqrt{2}}{\sqrt{v^2 + 2\sigma^2 u} + v}. \tag{15}$$

With the identification $\sigma^2 = 2D$, the result (9) is recovered.

For the binomial random walk, the survival probability F_n^+ is a highly non-trivial function of the velocity v, depending on whether v is rational or irrational, because of the underlying lattice structure [18]. In particular, the limit survival probability $F^+ = \lim_{n\to\infty} F_n^+$, which

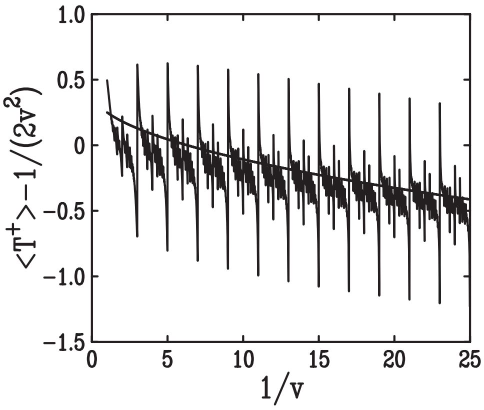

Figure 1. Plot of the difference between the mean occupation time $\langle T^+\rangle$ for the symmetric binomial random walk and its continuum-limit expression (18), against $1/v$. Thick curve: sum of last two terms of equation (17).

is non-zero for $v < 0$, is discontinuous at any rational value of v. The corresponding discontinuities are algebraic numbers which can be determined explicitly [18]. In the continuum limit ($v \to 0^-$), F^+ is simply given by $|v|\sqrt{2}/\sigma$.

In order to better understand the nature of the corrections to the continuum limit, we consider the simple case of the asymptotic mean occupation time $\langle T^+\rangle$. For the symmetric binomial random walk, and for fixed $v > 0$, we have

$$\langle T^+\rangle = \sum_{n=1}^{\infty}\langle I_n\rangle = \sum_{n=1}^{\infty} 2^{-n} \sum_{k=k_0(v)}^{n} \binom{n}{k} \tag{16}$$

with $k_0(v) = \mathrm{Int}(n(1+v)/2) + 1$, where $\mathrm{Int}(x)$ is the integer part of x, i.e. the largest integer less than or equal to x. As shown in the appendix, the behaviour of this expression as $v \to 0$ is given by

$$\langle T^+\rangle = \frac{1}{2v^2} + \frac{A}{\sqrt{v}} + \frac{5}{12} + \cdots \tag{17}$$

with

$$A = \sqrt{\frac{2}{\pi}}\,\zeta\left(-\frac{1}{2}\right) = -0.165\,869\,209.$$

The first term in (17) corresponds to the continuum-limit result (see equation (11))

$$\langle T^+\rangle_{\mathrm{Brown}} = \frac{1}{2v^2} \tag{18}$$

because $D = 1/2$ for the binomial random walk. The second term in (17) is surprising in several respects: it is of relative order $v^{3/2}$, instead of the naturally expected v^2, and the corresponding amplitude A is transcendental.

Figure 1 shows a plot of the difference between the exact value of $\langle T^+\rangle$, obtained by evaluating numerically (16), and its continuum-limit expression (18). The sum of the last two terms in (17), shown as a thick curve, correctly describes the mean asymptotic behaviour of the plotted quantity. Superimposed periodic oscillations, with period two, are also clearly visible. Similar oscillations, due to the lattice underlying the discrete walk, are also encountered when considering other quantities [18] (see especially figures 14 and 15 therein).

Consider finally the case where the steps have a Cauchy distribution

$$\rho(\eta) = \frac{1}{\pi(1+\eta^2)}.$$

Because of the stability of the Cauchy law, the probability $\langle I_n \rangle = \mathcal{P}(x_n > vn)$ is independent of n:

$$\langle I_n \rangle = \mathcal{P}(x_n > vn) = a(v) = \int_v^\infty \mathrm{d}u\, \rho(u) = \frac{1}{2} - \frac{1}{\pi} \arctan v. \tag{19}$$

Hereafter we denote this expression by a, for short. Using (13), we obtain

$$\sum_{n=0}^{\infty} F_n^+ z^n = (1-z)^{-a}$$

hence

$$F_n^+ = \frac{\Gamma(n+a)}{n!\,\Gamma(a)} \approx \frac{n^{-\theta(v)}}{\Gamma(a)} \qquad (n \gg 1)$$

with

$$\theta(v) = 1 - a = \frac{1}{2} + \frac{1}{\pi} \arctan v. \tag{20}$$

The survival probability F_n^+ falls off as a power law, with a continuous family of persistence exponents $\theta(v)$ [6–9, 19].

From equation (12), we obtain

$$\mathcal{P}\left(T_n^+ = k\right) = \frac{\Gamma(a+k)}{k!\,\Gamma(a)} \frac{\Gamma(n+1-k-a)}{(n-k)!\,\Gamma(1-a)} = \binom{n}{k} \frac{B\,(a+k, 1-a+n-k)}{B\,(a, 1-a)} \tag{21}$$

where the beta function is defined as

$$B(a,b) = \int_0^1 \mathrm{d}u\, u^{a-1}(1-u)^{b-1} = \frac{\Gamma\,(a)\,\Gamma\,(b)}{\Gamma\,(a+b)}.$$

Defining the β density as

$$\beta_{a,b}(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} \qquad (0 < x < 1),$$

one can rewrite (21) as

$$\mathcal{P}\left(T_n^+ = k\right) = \int_0^1 \mathrm{d}u \binom{n}{k} u^k\,(1-u)^{n-k}\, \beta_{a,1-a}(u). \tag{22}$$

In the continuum limit where n and k are simultaneously large, with a fixed ratio $x = k/n$, the binomial distribution inside (22) converges to $\delta(u-x)$, so the limiting probability density function of T_n^+/n reads

$$\lim_{n\to\infty} f_{T_n^+/n}(x) = \beta_{a,1-a}(x) = \frac{\sin \pi a}{\pi} x^{a-1}(1-x)^{-a}. \tag{23}$$

In particular, if $v = 0$, then $a = 1/2$, and one recovers the arcsine law (4).

To conclude, let us briefly consider the case of a moving boundary whose position obeys an arbitrary power law: $X(t) = w\, t^\nu$ [18, 20, 21].

If the steps have a narrow distribution, the continuum description can again be used in the regime of long times ($t \gg 1$) and weak bias ($|w| \ll 1$). A generalization of (14) shows that the mean occupation time reads

$$\langle T_t^+ \rangle = \int_0^t \mathrm{d}t' \langle I_{t'} \rangle = \frac{1}{2} \int_0^t \mathrm{d}t'\, \mathrm{erfc}\left(\frac{w}{\sigma\sqrt{2}} (t')^{\nu-1/2}\right).$$

The case $\nu = 1/2$ therefore demarcates between two regimes. For $\nu < 1/2$, the bias w is irrelevant, so $\langle T_t^+ \rangle \approx t/2$, and the arcsine law (4) still holds asymptotically. In contrast, for $\nu > 1/2$, any weak bias is relevant, so a non-trivial limiting law for the occupation time

T^+ is expected (for $w > 0$), generalizing (10), with $\langle T^+ \rangle \sim (\sigma/w)^{2/(2\nu-1)}$. In the marginal situation of a parabolic boundary ($\nu = 1/2$), there is a continuously varying persistence exponent $\theta(w)$ [20, 21], and the fraction T_t^+/t admits a non-trivial limit distribution, which also continuously depends on w.

If the steps have a symmetric broad (Lévy) distribution, with tails falling off as $\rho(\eta) \sim |\eta|^{-\mu-1}$, with $0 < \mu < 2$, the above discussion on the relevance of the bias w still applies, with the marginal situation being $\nu = 1/\mu$. The case of Cauchy-distributed steps in the presence of a ballistic boundary ($\mu = \nu = 1$) is an interesting example of this marginal situation, where the dependence on the bias $w = v$ of the persistence exponent (20) and of the limit distribution (23) are known explicitly.

Appendix. Expansion for $v \to 0$ of expression (16)

In this appendix we investigate the behaviour as $v \to 0$ of expression (16) of the mean occupation time $\langle T^+ \rangle$, i.e.

$$\langle T^+ \rangle = \sum_{n=1}^{\infty} \langle I_n \rangle = \sum_{n=1}^{\infty} 2^{-n} \sum_{k=k_0(v)}^{n} \binom{n}{k}.$$

We set $u = 1/v$, and introduce the Laplace transform $L(s) = \mathcal{L}_u \langle T^+ \rangle$. The above expression yields

$$L(s) = \frac{1}{s} \sum_{n=1}^{\infty} 2^{-n} \sum_{k=k_0}^{n} \binom{n}{k} \exp\left(-\frac{sn}{2k-n}\right)$$

with $k_0 = k_0(0) = \mathrm{Int}(n/2) + 1$.

Introducing the contour-integral representation

$$\binom{n}{k} = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{1}{z^{k+1}(1-z)^{n-k+1}}$$

where the contour encircles the origin, and summing over k at fixed $\ell = 2k - n \geqslant 1$, we obtain

$$L(s) = \frac{\mathrm{e}^{-s}}{s} \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \sum_{\ell=1}^{\infty} \frac{4}{(2z)^{\ell}\left(4z(1-z) - \mathrm{e}^{-2s/\ell}\right)}$$

hence, after some algebra,

$$L(s) = \frac{\mathrm{e}^{-s}}{s} \sum_{\ell=1}^{\infty} \frac{(1+W_\ell)^{-\ell}}{W_\ell} = \frac{\mathrm{e}^{s}}{s} \sum_{\ell=1}^{\infty} \frac{(1-W_\ell)^{\ell}}{W_\ell}$$

with $W_\ell = \sqrt{1 - \mathrm{e}^{-2s/\ell}}$.

In the regime of interest ($s \to 0$), we have

$$L(s) = L_0(s) + L_1(s) + \cdots$$

with

$$L_0(s) = \frac{1}{\sqrt{2s^3}} \sum_{\ell=1}^{\infty} \sqrt{\ell}\, \mathrm{e}^{-\sqrt{2s\ell}} \qquad L_1(s) = \sum_{\ell=1}^{\infty} \left(\frac{1}{2\sqrt{2s\ell}} - \frac{1}{6}\right) \mathrm{e}^{-\sqrt{2s\ell}}$$

and so on.

The leading series $L_0(s)$ has to be investigated in some detail, by means of its Mellin transform $M_0(x)$, for which we obtain a closed-form expression:

$$M_0(x) = \int_0^{\infty} \mathrm{d}s\, s^{x-1}\, L_0(s) = 2^{2-x}\, \Gamma(2x-3)\, \zeta(x-2) \qquad (\mathrm{Re}\, x > 3)$$

where ζ is Riemann's zeta function. The behaviour of the subleading series $L_1(s)$ can be estimated to leading order, replacing the sum over ℓ by an integral over $\sqrt{2s\ell}$. We thus obtain $L_1(s) \approx 1/(3s)$.

Inverting successively the Mellin and Laplace transforms, we obtain expression (17), i.e.

$$\langle T^+ \rangle = \frac{1}{2v^2} + \frac{A}{\sqrt{v}} + \frac{5}{12} + \cdots$$

with

$$A = \sqrt{\frac{2}{\pi}}\,\zeta\left(-\frac{1}{2}\right) = -0.165\,869\,209$$

and where the dots represent a contribution going to zero as $v \to 0$.

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